# CS170

## Linear Programming

# Linear programming

- What is linear?

  - Addition ($x_1 + x_2$)
  - Scalar multiplication ($a x_1$, $b x_2$)

- Optimization problems have 3 parts:

  1. Variables: $x_1, x_2, \dots x_n$.
  2. Objective function: some function of the variables that we're trying to minimize or maximize
  3. Constraints: Relationships (usually inequalities) the variables must satisfy

- Linear programming is a type of optimization problem with a linear objective function and linear constraints.

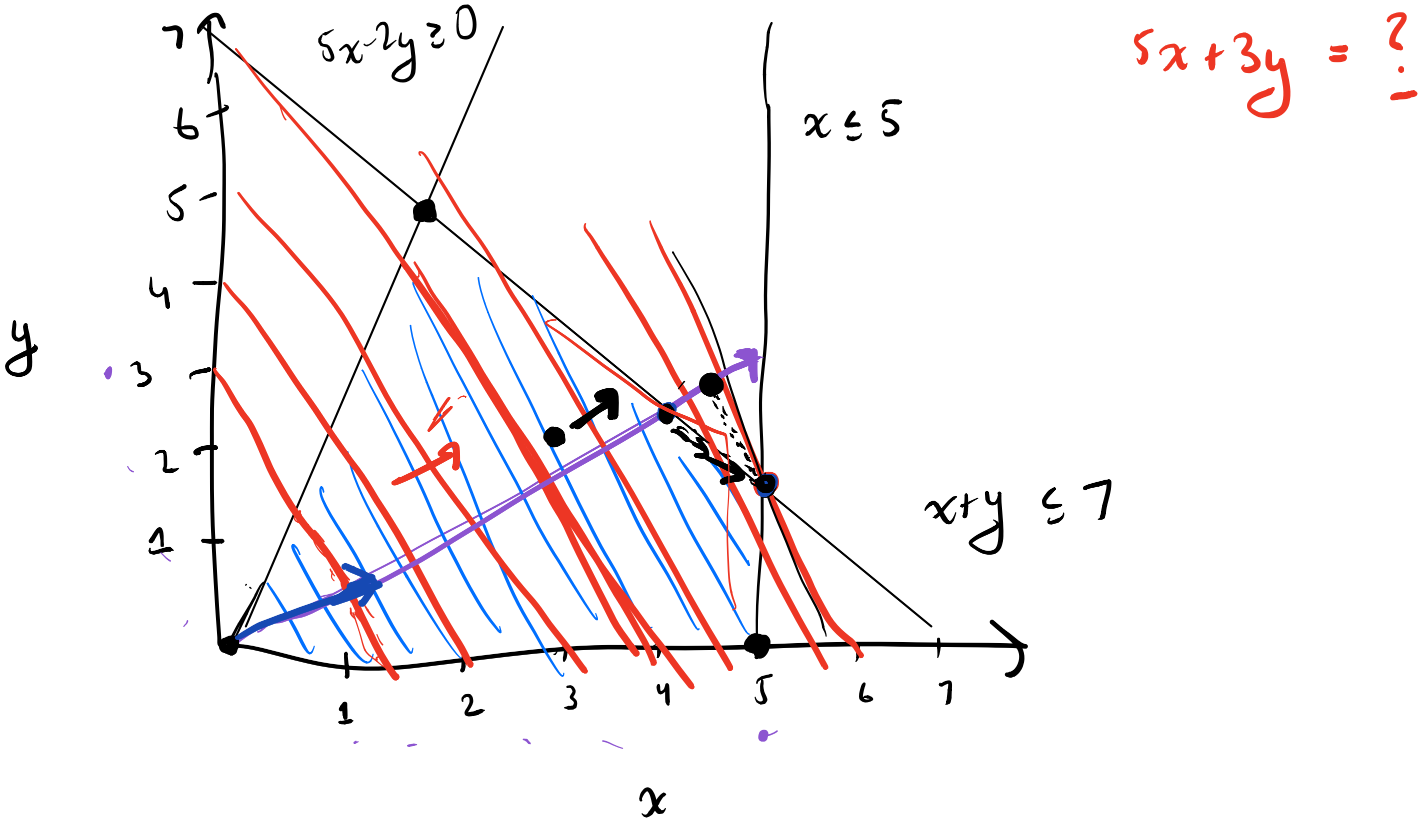
## Example

$$\max(5x + 3y) \quad \text{objective}$$

such that

$$\left.\begin{aligned} 5x - 2y &\geq 0 \\ x + y &\leq 7 \\ x &\leq 5 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}\right\} \text{constraints}$$

non-negativity constraints - don't forget! ($x \geq 0$, $y \geq 0$)




## How to solve?

- Max/min value is always at a <u>vertex</u>!
  (because of convexity - will cover later)

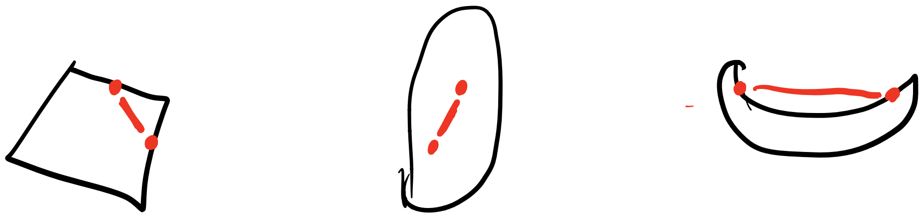
- Find all vertices and compare values

- How to see this? Look at level sets

Level set - set of points where value of objective is equal ( $3x+5y=c$, for some $c$)

## Convexity

- For a well-defined linear program, the set of feasible solutions will be convex.



convex      convex      non-convex

( Formally, if $\vec{x}$ and $\vec{y}$ are feasible solutions of the linear program, then $\forall \lambda \in [0, 1]$, so is $\vec{z} = \lambda \vec{x} + (1 - \lambda) \vec{y}$.

( intuitively - anything between them)

Because of this, one of three conditions must hold -

1. no feasible solutions ( $x \geq 1$, $x \leq -3$)
2. unbounded, with no optimum ( min $x_1 + x_2$, 0 $x_1 + x_2 \geq 0$)
3. vertex of convex polytope

Why vertex?

(1). For any point in the interior, can do better (be more efficient with constraints)

(2) For any point on the edge, either the edge is optimal, or can move towards a more optimal direction.

Simplex!

next week - duality. Very important - two ways of looking at linear programs, by flipping the constraints and variables

# Linear algebra LPs

Since constraints and objective functions are linear, can write constraints & inequalities with vectors and matrices.

Eg. $\max(5x+3y)$

such that
$$5x-2y \geq 0$$
$$x+y \leq 7$$
$$x \leq 5$$
$$x \geq 0$$
$$y \geq 0$$

$=$ $\max \quad c^T \vec{z}$

such that
$$A\vec{z} \geq b$$
$$\vec{z} \geq 0$$

where $\vec{c} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, $\vec{z} = \begin{bmatrix} x \\ y \end{bmatrix}$, $A = \begin{bmatrix} 5 & -2 \\ -1 & -1 \end{bmatrix}$

$$\vec{b} = \begin{bmatrix} 0 \\ 7 \end{bmatrix}$$

*Note*: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

# 1 LP Basics

**Linear Program.** A *linear program* is an optimization problem that seeks the optimal assignment for a linear objective over linear constraints. Let $x \in \mathbb{R}^d$ be the set of variables and $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$. The canonical form of a linear program is

$$\begin{aligned} &\text{minimize } c^\top x \\ &\text{subject to } \boxed{Ax \geq b} \\ &\underline{x \geq 0} \end{aligned}$$

$\hat{A} = \begin{bmatrix} A \\ -A \end{bmatrix}$ $\hat{b} = \begin{bmatrix} b \\ -b \end{bmatrix}$

$\hat{A} x \geq \hat{b}$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Any linear program can be written in canonical form.

Let's check this is the case:

(i) What if the objective is maximization? $c \to -c$

(ii) What if you have a constraint $Ax \leq b$? $-Ax \geq -b$ $A \to -A$, $b \to -b$

(iii) What about $\underline{Ax = b}$? $Ax \geq b$, $-Ax \geq -b$

(iv) What if the constraint is $x \leq 0$? $z = -x$ $x \to -z$, $z \geq 0$

(v) What about unconstrainted variables $x \in \mathbb{R}$? $x_i$

$\boxed{0, 5} \to x = -5$ $x \to (x^+ - x^-)$

$2, 7$ $x_i^+$ $x_i^-$ $x^+, x^- \geq 0$

$3, 8$

**Dual.** The dual of the canonical LP is

$$\begin{aligned} &\text{maximize } b^\top y \\ &\text{subject to } A^T y \leq c \\ &y \geq 0 \end{aligned}$$

**Weak duality**: The objective value of any feasible dual $\leq$ objective value of any feasible primal
**Strong duality**: The *optimal* objective values of these two are equal.
Both are solvable in polynomial time by the Ellipsoid or Interior Point Method.

## 2 Job Assignment

There are $I$ people available to work $J$ jobs. The value of person $i$ working 1 day at job $j$ is $a_{ij}$ for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Each job is completed after the sum of the time of all workers spend on it add up to be 1 day, though partial completion still has value (i.e. person $i$ working $c$ portion of a day on job $j$ is worth $a_{ij}c$). The problem is to find an optimal assignment of jobs for each person for one day such that the total value created by everyone working is optimized. No additional value comes from working on a job after it has been completed.

(a) What variables should we optimize over? I.e. in the canonical linear programming definition, what is $x$?

$x_{ij} = 1$ $\quad x_{ij}$ - time worker $i$ spends on job $j$

$i' \neq i$, $j' \neq j$, $x_{ij'} = 0$

1 $\quad x_{ij} = 0.2$ worker $i$ spends 20% on job $j$

$x_{i'j} > 0$

(b) What are the constraints we need to consider? Hint: there are three major types.

$x_{ij} \geq 0$ $\qquad \forall j \ \underline{\sum_i x_{ij} \leq 1}$.

$\forall i \ \underline{\sum_j x_{ij} \leq 1}$

(c) What is the maximization function we are seeking?

$$\sum_{i,j} x_{ij} \cdot a_{ij}$$

## 3 Linear regression

In this problem, we show that linear programming can handle linear regression. Let $A \in \mathbb{R}^{n \times d}$ and $b \in \mathbb{R}^d$ be given, where $n, d$ are not assumed to be constant. However, assume all input numbers have constant bits.

(a) Recall that the $\ell_1$ norm of a vector $v$ is given by $\|v\|_1 = \sum_{i=1}^d |v_i|$. The L1 regression problem asks you to find $x \in \mathbb{R}^d$ that minimizes $\|Ax - b\|_1$.

(i) Provide a linear program that finds the opimtal $x$, given $A, b$.

(ii) Argue that it can be solved in polynomial time (in $n, d$).

(b) Recall that the $\ell_\infty$ norm of a vector $v$ is given by $\|v\|_\infty = \max_i |v_i|$. The $L_\infty$ regression problem asks you to find $x \in \mathbb{R}^d$ that minimizes $\|Ax - b\|_\infty$.

(i) Provide a linear program that finds the opimtal $x$, given $A, b$.

(ii) Argue that it can be solved in polynomial time (in $n, d$).

## 4 Provably Optimal

Consider the following linear program:

$$\begin{aligned} \max\ & x_1 - 2x_3 \\ & x_1 - x_2 \leq 1 \\ & 2x_2 - x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

For the linear program above,

(a) First compute the dual of the above linear program

(b) show that the solution $(x_1, x_2, x_3) = (3/2, 1/2, 0)$ is optimal **using its dual**. You do not have to solve for the optimum of the dual. (*Hint:* Recall that any feasible solution of the dual is an upper bound on any feasible solution of the primal)

## 5 Taking a Dual

Consider the following linear program:

$$\begin{aligned} \max\ & 4x_1 + 7x_2 \\ & x_1 + 2x_2 \leq 10 \\ & 3x_1 + x_2 \leq 14 \\ & 2x_1 + 3x_2 \leq 11 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Construct the dual of the above linear program.