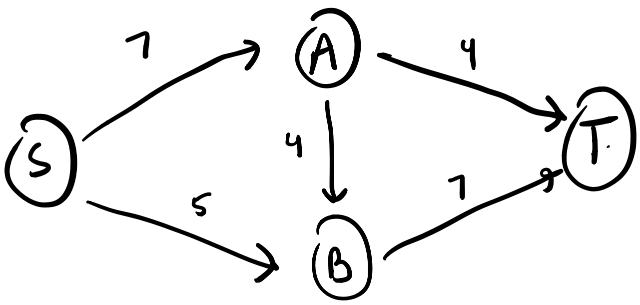
CS170

SECTION 9

FLOW, DUALITY, ZSGs

Network Flow

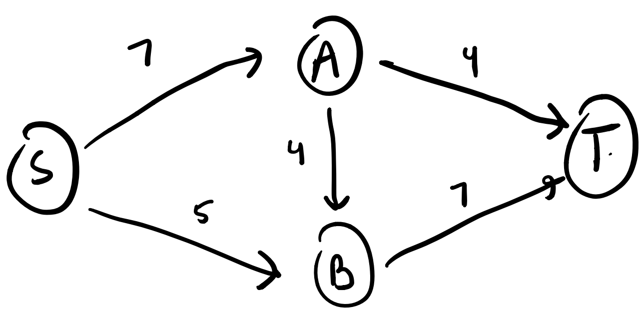


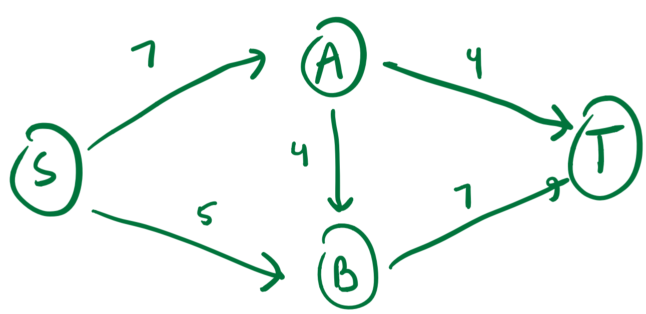
Max flow we can push from s to t, given capacity constraints

Ford-Fulkerson algorithm

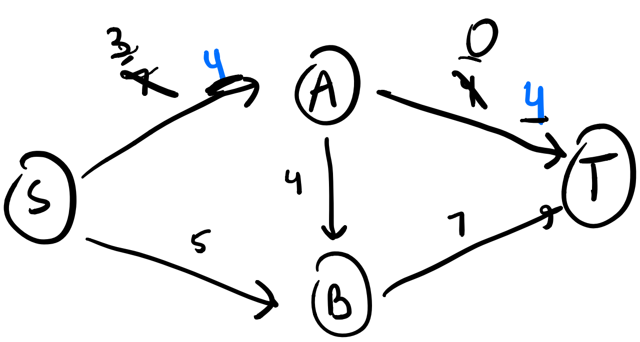
1. While there is a path from s-t in the residual graph, fill that path as much as possible

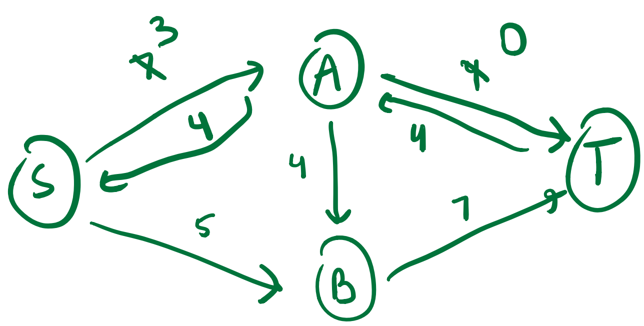
2. Update residual graph



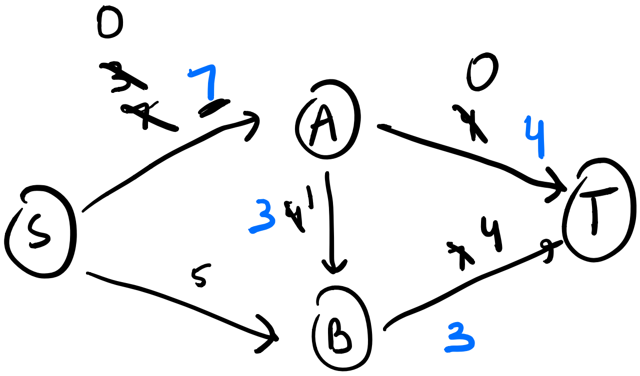


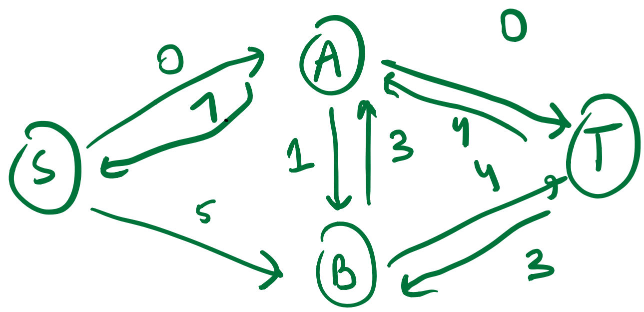
(Send flow along $S \to A \to T$ path)



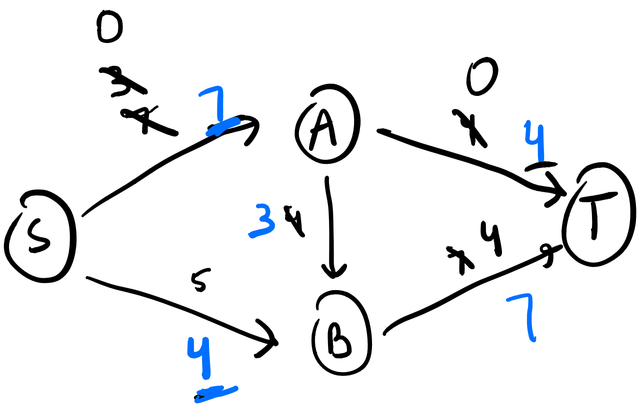


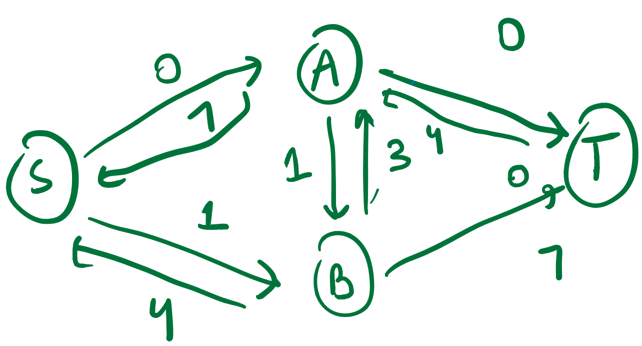
(send flow along S → A → B → T path)





(send flow along S → B → T path)





No flow can pass from s → t anymore in the residual graph! So we are done!

(Have also found the min-cut!)

max z

Duality

In LP, we try to maximize some variables, subject to certain constraints (variables can't be too large)

In the dual, we pretend to be the constraints! We place an upper bound on the value of the objective using the constraints. (But constraints can't be too strict!)

Example:

$$\max \ \boxed{P + 2B}$$

$$2\,(P \le 400)$$
$$+\ 3\,(B \le 300)$$
$$2P + 3B \le 1200$$

$$P, B \ge 0$$

$$P + 2B \le \ldots$$

Vector form:

$$\max \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} P \\ B \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} P \\ B \end{bmatrix} \le \begin{bmatrix} 400 \\ 300 \\ 1200 \end{bmatrix}$$

$$\begin{bmatrix} P \\ B \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

New variables are now coefficients of the constraints!

Want to choose coefficients so we can get an inequality that contains the objective (so we can bound it)

$$\max \; P + 2B$$

$$x \cdot (P \le 400)$$
$$y \cdot (B \le 300)$$
$$z \cdot (2P + 3B \le 1200)$$
$$P, B \ge 0$$

variables are how much we scale each of the inequalities before we add them together.

$x < 10$

x'

$\boxed{x'} \ge x$

$x' < 10$

$x < 10$

$$x \ge 0$$
$$y \ge 0$$
$$z \ge 0$$

$$Px \le 400x$$
$$By \le 300y$$
$$+ \; 2Pz + 3Bz \le 1200z$$

$$(x + 2z)P + (y + 3z)B \le \{400x + 300y + 1200z\}$$

$x + 2z \ge 1$

$y + 3z \ge 2$

(Pattern match!)

$x + 2z \ge 1$ (inequality is an even better upper bound)

coefficient of P in the primal objective

$y + 3z \ge 2 \rightarrow$ coefficient of B in the primal objective

Want to minimize the upper bound!

(constraints want to make primal objective small)

$$\min \; 400x + 300y + 1200z$$

subject to
$$\left.\begin{aligned} x + 2z &\geq 1 \\ y + 3z &\geq 2 \end{aligned}\right\}$$
$$x \geq 0$$
$$y \geq 0$$
$$z \geq 0$$

Dual

Vector formulation comparison:

Vector form primal

$$\max \begin{bmatrix} 1 \\ 2 \end{bmatrix}^T \begin{bmatrix} P \\ B \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} P \\ B \end{bmatrix} \leq \begin{bmatrix} 400 \\ 300 \\ 1200 \end{bmatrix}$$

$$\begin{bmatrix} P \\ B \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Vector form dual

$$\min \begin{bmatrix} 400 \\ 300 \\ 1200 \end{bmatrix}^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \geq 0$$

ZSGs

- 2 players:
  - Maximizer, tries to make score more positive
  - Minimizer, tries to make score more negative
- Payoff matrix – grid of options
  - Minimizer – columns
  - Maximizer – rows

	m	t
e	3	-1
s	-2	1

- Each player wants to choose the optimal probability distribution to get the highest/lowest score
- Expected payoff: $\sum_{i,j} G_{ij} \cdot \text{Prob}[$ Row plays i, Column plays j $]$
- Assume that the Row player announces their distro first. Then the column player will choose the option that minimizes

expected payoff:

		m	t
x_1	e	3	-1
x_2	s	-2	1

$x_1 + x_2 = 1$

0.6 0.4

Payoff: $\min\{3x_1, -2x_2, -x_1 + x_2\}$

Row player wants to maximize this!

$$\max_{x_1, x_2}\left(\min\{3x_1 - 2x_2, -x_1 + x_2\}\right)$$

Very cool fact: Using this same logic for the column player gives you the dual! [shows there's an optimal value]

$$\max \quad z$$

$$z \le 3x_1 - 2x_2$$

$$z \le -x_1 + x_2$$

$$x_1 + x_2 = 1$$

$$x_1, x_2 \ge 0$$